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Conserved energy-momentum as a class of tensor densities

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Abstract. A concept is presented where the conserved energy-momentum tensor is not fixed uniquely. The addition of identically conserved tensors, which can be generated by adding surface terms to a Lagrangian is viewed as a kind of gauge freedom which does not affect the predictions for experiments. The canonical energy-momentum tensor obtained from the Noether theorem is used to define a class of tensors which contains practically all known conserved tensors, especially for Einstein's theory of general relativity. But it also applies to situations without full Poincaré symmetry, where other concepts fail. The main tool is the Belinfante–Rosenfeld construction of a gravitational energy-momentum tensor from the tetrad Lagrangian of general relativity; which turns out to be precisely a multiple of the Einstein tensor. The canonical energy-momentum tensor obtained from the tetrad Lagrangian is analysed. It is shown that periodic solutions of the Einstein field equations have zero mean energy. The energy density of static solutions in vacuum is negative under quite general conditions, whereas the total energy turns out to be positive through agreement with the Arnowitt–Deser–Misner surface integral.

1. Introduction

The concept of energy-momentum as a local, conserved tensor with positive energy density can be considered well understood as long as gravitation is not included. However, in general relativity theory there is a well known argument leading to the conclusion that no covariant local energy-momentum tensor exists for gravity [1, p 467]. If a covariant tensor is quadratic in the first derivatives of the metric, one can transform it at any point to local Lorentz coordinates in which the first derivatives vanish, implying that the tensor is zero everywhere. Many attempts have been made to escape from this argument by weakening the conditions on the desired energy-momentum tensor.

From Maxwell theory we have two arguments in favour of an energy-momentum tensor containing only first derivatives. The first is that $U(1)$-gauge invariance requires that the energy-momentum tensor depends only on $F_{\mu\nu}$, the Maxwell field tensor; any tensor which contains second derivatives of the potential A_μ is not gauge invariant. This originates in the additive nature of the $U(1)$-gauge transformation

$$A_\mu \to A_\mu + \partial_\mu f. \tag{1}$$

But in Einstein's nonlinear theory of gravity we do not have this kind of additive gauge transformations that restrict the form of the energy-momentum tensor. The Ricci tensor R_{kl} and the Einstein tensor $G_{kl} = R_{kl} - \frac{1}{2}g_{kl}R$ both are covariant and contain second derivatives. The second argument is the behaviour of the densities at spatial infinity. For fields that decay at $1/r$ at infinity a density quadratic in the first derivatives is always integrable at infinity. But as can be seen from the Ricci and Einstein tensors, the field equations guarantee integrability at infinity for second-derivative tensors as well.

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So these clearcut arguments fail for the construction of a gravitational energy-momentum tensor, moreover no such tensor with positive definite energy density seems to exist. So the basic idea is to follow the structure of the electromagnetic theory [1,5]. In analogy to the equations

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \partial_\nu j^\nu = 0 \tag{2}$$

where charge conservation is guaranteed through the antisymmetry of the electromagnetic field tensor F, in general relativity a linearized part, conserved due to the linearized Einstein equations, is split off the Einstein tensor. This then serves as one part of the Einstein field equations, and the rest is reckoned as a nonlinear source[1]. The Einstein field equations $G^{kl} = \gamma T^{kl}$ are written as

$$\partial_i H^{ikl} = \gamma(T^{kl} + t^{kl}) \tag{3}$$

with

$$t^{kl} = \frac{1}{\gamma}(-G^{kl} + \partial_i H^{ikl}) \tag{4}$$

where the antisymmetry of H^{ikl} in the indices i, k leads to a vanishing divergence

$$\partial_k \partial_i H^{ikl} = 0. \tag{5}$$

One arrives at the pseudo energy-momentum tensor t^{kl} for gravity which is not covariant. But this way is criticized for its arbitrariness in the nonlinear part of the subtraction, leading to an infinite number of possible tensors [1, section 20.3].

We may also consider covariant tensors containing second derivatives of the metric. But when we try to preserve general covariance, we are invariably led to a null result: total energy-momentum vanishes. This was recently demonstrated by Gotay and Marsden [2]. They generalized the Belinfante–Rosenfeld (BR) [3, 4] construction from special relativity to general relativity. In the special relativistic construction, the Poincaré invariance of a Lagrangian is exploited to make the conserved canonical energy-momentum tensor symmetric by adding a spin part. The BR symmetrized energy-momentum tensors for matter and electromagnetic field are the generally accepted ones [5, 6]. When gravity is included these BR-tensors may also be derived [5, 6] by varying the matter and electromagnetic field Lagrangians with respect to the metric; these tensors are called Hilbert tensors [2]. In the case of gravity this leads to a tensor which vanishes on-shell, i.e. when the field equations are satisfied [5], since the Hilbert tensor for gravity is the Einstein tensor.

Gotay and Marsden [2] constructed a fully general covariant generalization of the special relativistic procedure, with precisely the same result, namely that the Einstein tensor should be the energy-momentum tensor for gravity. But since the null result implies that the total energy density is not positive definite, with a gravitational part that is always negative or zero, Gotay and Marsden suggested leaving out the gravitational part of the construction, as did Landau and Lifschitz [5]. This again leaves us without a systematic procedure for gravitation.

But it is remarkable that the generalized BR construction still works without the gravitational part of the Lagrangian. The reason is that the general covariant method leads to tensors with vanishing general covariant divergence for each part of the Lagrangian [7], but then this does not represent a conservation law. As is well known the vanishing of the general covariant divergence of the particle Hilbert tensor is equivalent to the geodesic equations, and holds even when particle energy-momentum is not conserved.

Recently Bak *et al* [8] applied the original special relativistic construction of Belinfante [3] to the Einstein–Hilbert action defined on $\boldsymbol{R}^4$, where general covariance includes Poincaré invariance, and came up with a nonzero result. They claimed that this was due to them using a first-derivative form of the Einstein–Hilbert Lagrangian, which is not scalar under general coordinate transformations, so that the covariant result by Gotay and Marsden [2] should not be applicable. We find that the result by Bak *et al* [8] is due only to their specific choice of a surface term added to the Lagrangian. We show that when we start from the pure Vierbein or tetrad Lagrangian of gravity [11], then the special relativistic gravitational field BR tensor is identical with the Einstein tensor density $-(8\pi K)^{-1}\sqrt{-g}G^k{}_l$, and the gravitational field equations are equivalent to the vanishing of the combined BR tensor for matter, electromagnetic field, and gravitation. Since our Lagrangian differs from the one Bak *et al* used by a surface term only, we conclude that the tensor they suggested (their (32)) stems from the surface term alone. Since the surface term cannot be fixed by integrability arguments, and the BR tensor depends on the choice of such a term, we are again open to arbitrariness.

We now propose to take this arbitrariness seriously. We have found discussions on the merits of different constructions for an energy-momentum tensor, but there are no arguments that any construction is ill-conceived in the way that it contradicts with experiment. In the case of gravity the theoretical criteria are positivity, symmetry and covariance, and they cannot all be satisfied in one tensor. So we suggest that uniqueness for the conserved energy-momentum tensor is not enforced, viewing it is a quantity which cannot be measured directly. A classical example for such a kind of quantity is the electromagnetic four-potential. We may choose any gauge for the calculation of the measurable physical quantities, the electric and magnetic fields. With respect to the energy and momentum densities, we also argue that their values cannot be measured directly. Only kinematic quantities, positions and velocities of positions of particles, and their changes in time, such as electric currents, are measured. We measure wavelength or frequencies of photons, of the electric currents they induce. From these kinematical quantities the energy is inferred on the basis of some theoretical formula. In order to get values, we always have to make assumptions about certain quantities, for example the gauge of the electromagnetic field or the coordinate system. Different assumptions then lead to different values. Once these assumptions have been fixed, we may use energy-momentum conservation to predict the final values of the kinematical quantities from their given initial ones, with the same result for any allowed choice of gauges etc. We can uniquely predict the count of a Wattmeter or photomultiplier, with defining a unique energy-momentum tensor.

On the other hand, the sources of the fields must be fixed uniquely since we could not otherwise predict the strength of the fields generated by them and the forces exerted on other particles or fields that enter the kinematical equations of motion. The sources are fixed once a Lagrangian is defined up to surface terms, since the Euler–Lagrange equations do not depend on surface terms. The Lagrangian itself need not be defined uniquely in order to describe a unique physical situation. We illustrate this with two rather well known and simple examples from nonrelativistic and relativistic mechanics.

First, let us consider the Lagrangian for a classical free nonrelativistic point particle,

$$L = \tfrac{1}{2}m\dot{x}(t)^2 \tag{6}$$

which yields upon variation with respect to $x(t)$ the Euler–Lagrange equation

$$m\ddot{x} = 0. \tag{7}$$

This equation of motion is invariant under the Galilei transformation

$$x \to x' = x - vt \tag{8}$$

whereas the Lagrangian changes by a boundary term:

$$L \to L' \frac{1}{2} m\dot{x}'^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(mxv + \frac{1}{2}mv^2 t\right). \tag{9}$$

Since the variation is carried out with fixed boundary values for $x(t)$, the equation of motion is not affected. When we apply the Noether theorem in its simplest form, exploiting the independence of the Lagrangian of the coordinate x, we obtain the conserved momentum $p = \partial L/\partial \dot{x}$ in a different form from L and L':

$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p' &= \frac{\partial L}{\partial \dot{x}'} = m(\dot{x}' + v). \end{aligned} \tag{10}$$

So the functional forms of the conserved momenta are different for the different coordinate systems, but we may use any of then in order to predict the motion of the particle in configuration space. When initial conditions for x and $\dot{x}$ are given, the final x and $\dot{x}$ will be the same, independent of the form of the conserved momentum we employ in the calculation. This is an example of nonuniqueness which is directly related to energy momentum. A similar example is provided by the gauge-variant Lagrangian for a classical relativistic point particle in an external electromagnetic field, given by

$$L = \tfrac{1}{2} m\dot{x}^\mu \dot{x}_\mu + q\dot{x}^\mu A_\mu. \tag{11}$$

Here the equations of motion are invariant under the gauge transformation (1), but the Lagrangian transforms as

$$L \to L' = L - \frac{\mathrm{d}}{\mathrm{d}\tau}(qf). \tag{12}$$

So we may use as generalized energy-momentum variables

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m\dot{x}_\mu + q(A_\mu + \partial_\mu f) \tag{13}$$

with arbitrary f, without losing energy-momentum conservation, and with identical predictions for the particle motion in configuration space.

In general a derivation of conserved energy momentum from a Lagrangian will depend on surface terms and so will not be unique, as is shown in this article for the BR construction. So we take the attitude that it need not be unique, just as the Lagrangian itself, since we do not need its uniqueness to arrive at unique predictions for the outcome of experiments.

The formal concept is to view energy momentum as a quantity which is not represented by a single tensor or tensor density, but by a whole class of such objects, where two elements of the class may differ by an identically conserved tensor. We use the term 'identically conserved' for any current, the divergence of which vanishes as a functional of the fields, irrespective of whether the field equations are satisfied or not. In this way we take surface terms in the Lagrangian into account, and need not fix the Lagrangian uniquely. Accepting such a concept obviously implies that the concept of an absolute, positive definite energy density is given up. We feel free to take this step since so far no positive energy density for gravity have been proposed. We build the class of conserved tensors from the canonical energy-momentum tensor (CEM), as obtained from the Noether theorem. This is very important, since in situations where we do not have the full Poincaré symmetry or full general covariance, such as for particles in external fields, the CEM tensor or parts of it may be the only conserved currents we have at all. This situation is ubiquitous, but rarely discussed in the context of other concepts.

This article is structured in the following way. In section 2 we look at the special relativistic BR construction, its ambiguity, and prove the consistency between the gravitational field equations and the conservation law. The BR construction is used to split the Einstein tensor density into a canonical part and an identically conserved spin part. In principle, this is the same idea as that behind the construction of pseudo energy-momentum tensors, and these are shown to belong to the same class as the CEM tensor. We use orthonormal tetrad, or Vierbien fields to derive the results in section 2 and show that the BR tensor is the same whether constructed from tetrad fields or from the metric. Our central result is derived under general conditions on a first-derivative action for the Vierbein fields and matter, applicable to the Einstein–Hilbert action modified by a surface term, and coupled to matter. We show that the tensor given by Bak *et al* [8] stems from surface terms alone. Spin-$\frac{1}{2}$ matter is included in space-time without torsion, and the electromagnetic field is treated separately.

In section 3, the roles of the CEM and identically conserved tensors are elucidated. First we discuss formal aspects (section 3.1), then the various cases matter and fields, individually or together, and show that, for test particles in external fields (section 3.2), the currents of the CEM tensor can be the only conserved quantities available. We show that the position of the indices and the factor $\sqrt{-g}$ is decisive, and that the CEM tensor in this case is not identical with the matter part of the source of gravitation. The discussion is extended to quantum matter with spin-0 or spin-$\frac{1}{2}$ in external fields (section 3.3), and it is shown that the BR tensor need not be symmetric since we need not have full Poincaré symmetry in this case. Next, electromagnetic fields (section 3.4) and gravitational fields (section 3.5) are included as dynamical fields. It is shown in section 3.5.1 that the Einstein field equations are equivalent to the vanishing of the combined BR tensor for matter and fields. Finally, in section 3.5.2, the CEM tensor for gravity is analysed with respect to positivity for periodic and static solutions of the Einstein field equations. We give a summary in section 4.

2. Tetrad Lagrangian and Belinfante–Rosenfeld tensor

2.1. Preliminaries

We consider $\boldsymbol{R}^4$ with a tetrad field $\boldsymbol{e}_a$, $a = 0, 1, 2, 3$ which forms a local basis of the tangent space in each point of $\boldsymbol{R}^4$

$$\boldsymbol{e}_a = a_a{}^k \partial_k. \tag{14}$$

We use $a, b, c, d, \ldots$ for tetrad indices, coordinate indices are denoted by $i, j, k, l, \ldots$. The dual θ_a, $a = 0, 1, 2, 3$ of the tetrad field (14) forms a local basis of the cotangent space with

$$\theta^a = e^a{}_k \, \mathrm{d}x^k \tag{15}$$

where duality is equivalent to the following relation for the coefficients

$$e_a{}^k e^b{}_k = \delta^a_b \qquad e_a{}^k e^a{}_l = \delta^k_l. \tag{16}$$

A tetrad field always induces a metric

$$g_{kl} = \eta_{ab} e^a{}_k e^b{}_l \tag{17}$$

where η_{ab} is the Minkowski metric, for which we use the Landau–Lifschitz [5] sign convention $+ - - -$. The tetrad field is orthonormal with respect to the metric,

$$e_a{}^k e_b{}^l g_{kl} = \eta_{ab}. \tag{18}$$

Given a metric we cannot necessarily find an orthonormal tetrad field such that (18) holds, a topological criterion has to be satisfied, namely that the second Stiefel–Whitney class vanishes [9]. We assume that it is satisfied, especially since only in this case a spin connection exists, which is indispensable for the description spin-$\frac{1}{2}$ matter.

A linear connection is determined by a matrix $\omega^a{}_b$ of 1-forms,

$$\omega^a{}_b = \gamma^a{}_{bc}\theta^c \tag{19}$$

with coefficients $\gamma^a{}_{bc}$. We only consider the case where the tetrad field induces a Riemannian structure of $\boldsymbol{R}^4$, which requires that the torsion $\Sigma^a = \mathrm{d}\theta^a + \omega^a{}_b \wedge \theta^b$ vanishes, In this case the holonomic components $\Gamma^i{}_{jk}$ of the connection reduce to the Christoffel symbols derived from the induced metric (17), and the coefficients $\gamma^a{}_{bc}$ can be expressed through the anholonomy $\Omega^a{}_{bc}$ [11], which is defined by

$$\mathrm{d}\theta^a = \tfrac{1}{2}\Omega^a{}_{bc}\theta^b \wedge \theta^c. \tag{20}$$

We have [9–11]

$$\Omega^a{}_{bc} = e_b{}^j e_c{}^k \partial_j e^a{}_k - e_c{}^j e_b{}^k \partial_j e^a{}_k \tag{21}$$

and

$$\gamma^a{}_{bc} = \tfrac{1}{2}(\Omega_{bc}{}^a - \Omega_c{}^a{}_b + \Omega^a{}_{bc}). \tag{22}$$

Tetrad indices are raised and lowered with η, coordinate indices with g.

A spin structure is defined through the spin connection

$$\sigma = -\tfrac{1}{4}\omega^a{}_b\gamma_a\gamma^b \tag{23}$$

where γ_a are the Dirac gamma-matrices with

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}I. \tag{24}$$

The covariant derivative of a Dirac spinor $\psi_\alpha, \alpha = 1, 2, 3, 4$ is given by [9, 10]

$$\mathrm{D}\psi_\alpha = \mathrm{d}\psi_\alpha - \sigma_\alpha{}^\beta\psi_\beta = \theta^c\mathrm{D}_c\psi_\alpha. \tag{25}$$

We explicitly have

$$\mathrm{D}_c\psi = e_c{}^k\partial_k\psi + \tfrac{1}{4}\gamma^a{}_{bc}\gamma_a\gamma^b\psi \tag{26}$$

which depends on the derivatives of the tetrad field, but only via the anholonomy Ω. It is a remarkable fact that this general-covariant structure is at the same time a local Lorentz gauge-covariant structure, as was shown by Utiyama [14]. The scalar curvature R is at the same time the first-order Lagrangian for the Lorentz gauge fields $\gamma^a{}_{bk} = \gamma^a{}_{bc}e^c{}_k$. Local Lorentz gauge transformations affect only the tetrad and spin indices,

$$e_a{}^k \to \Lambda(x)_a{}^b e_b{}^k \qquad \psi_\alpha \to S(\Lambda(x))_\alpha{}^\beta\psi_\beta \tag{27}$$

where $S(\Lambda)$ is the representation of the Lorentz group on the spinor space. The metric is gauge invariant.

2.2. *Belinfante–Rosenfeld construction*

We start from the Einstein–Hilbert action

$$S_{EH} = \frac{c^3}{16\pi\kappa}\int \mathrm{d}^4x\,\sqrt{-g}R \tag{28}$$

which contains second derivatives of the metric. These second derivatives can be eliminated by adding a surface term which brings it into the first-order form $\bar{S}_{EH} = \frac{c^3}{16\pi\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\bar{R}$ with

$$\bar{R} = g^{kl}(\Gamma^i{}_{kj}\Gamma^j{}_{li} - \Gamma^i{}_{kl}\Gamma^j{}_{ij}) \tag{29}$$

as used by Bak *et al* [8]. Another possibility is to first express the metric by the tetrad field and then eliminate the resulting second derivatives of the tetrad field by adding some surface term to obtain [11]

$$\tilde{R} = \Omega^c{}_{ac}\Omega^{ba}{}_b - \tfrac{1}{4}\Omega^a{}_{bc}\Omega_a{}^{bc} - \frac{1}{2}\Omega^a{}_{bc}\Omega^b{}_a{}^c \tag{30}$$

and the corresponding action

$$\tilde{S}_{EH} = \frac{c^3}{16\pi\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\tilde{R}. \tag{31}$$

In this form $\tilde{R}$ contains only first derivatives, and resembles the one of the Lagrangian for the electromagnetic field; it is well suited to our purposes, it is also a common starting point for the construction of a Hamiltonian formulation of gravity [11].

We include matter either as classical point particles with action

$$S_{M,cl} = \frac{m}{2}\int \mathrm{d}\tau\,\dot{x}^k(\tau)\dot{x}^l(\tau)g_{kl}(x(\tau)) \tag{32}$$

or as quantum matter with action

$$S_{M,qm} = \begin{cases} \dfrac{1}{2m}\displaystyle\int \mathrm{d}^4x\; e[m\phi\phi^* - \eta^{ab}(e_a{}^k\partial_k\phi)(e_b{}^l\partial_l\phi^*)] & \text{(spin 0)} \\ \displaystyle\int \mathrm{d}^4x\; e\Re(mc\bar{\psi}\psi - \bar{\psi}\gamma^a \mathrm{i}\hbar \mathrm{D}_a\psi) & \text{(spin } \tfrac{1}{2}) \end{cases} \tag{33}$$

where $e = \sqrt{-g} = \det e^a{}_k$, and $\Re$ denotes the real part. Equations (31) and (33) together yield an action of the general form

$$S = \int \mathrm{d}^4x\; e\Lambda(\Omega^a{}_{bc}, \Phi, e_a{}^k\partial_k\Phi) \tag{34}$$

where Φ is any multicomponent matter field with all components transforming as scalars under coordinate transformations, only local Lorentz gauge transformations may affect the spin indices.

From (34) the Lagrangian has the form

$$\mathcal{L}(e_a{}^k, \partial_l e_a{}^k, \Phi, \partial_k\Phi) = e\Lambda(\Omega^a{}_{bc}, \Phi, e_a{}^k\partial_k\Phi). \tag{35}$$

We consider the potentials $\Omega^a{}_{bc}$ as dependent quantities, always expressed through the tetrad field as

$$\Omega^a{}_{bc} = -e^a{}_k(e_b{}^j\partial_j e_c{}^k - e_c{}^j\partial_j e_b{}^k) \tag{36}$$

which is derived from (21) with the help of $e^a{}_k\partial_j e_c{}^k = -e_c{}^k\partial_j e^a{}_k$. We note that even in the case of spin-$\frac{1}{2}$ matter we are not forced to give up this dependence. It is well known

[4,12], and will be derived below, that we get consistent field equations with a symmetric source tensor even in this case.

We now look at the BR construction, We use the Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots$ to indicate the special relativistic nature of the calculation, they are raised and lowered with η. For a Lagrangian $\mathcal{L}(e_a{}^k, \partial_\mu e_a{}^k) = \mathcal{L}'(g^{ij}, \partial_k g^{ij})$, with g and e related by (17), the BR tensor is given by [3, 4, 6, 8]

$$T_{BR}{}^{\mu\nu} = \eta^{\nu\nu'} T_C{}^\mu{}_{\nu'} - \partial_\alpha h^{\mu\nu\alpha} \tag{37}$$

where $T_C{}^\mu{}_\nu$ is the CEM tensor

$$T_C{}^\mu{}_\nu = \delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu e_a{}^k)} \partial_\nu e_a{}^k = \delta^\mu{}_\nu \mathcal{L}' - \frac{\partial \mathcal{L}'}{\partial(\partial_\mu g^{ij})} \partial_\nu g^{ij} \tag{38}$$

and the contribution $h^{\mu\nu\alpha}$ from spin that enters in (37) is defined by

$$h^{\mu\nu\alpha} = \tfrac{1}{2}(L^{\mu\nu\alpha} - L^{\alpha\nu\mu} - L^{\nu\alpha\mu}) \tag{39}$$

with

$$L^{\mu\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu e_a{}^k)} (S^{\alpha\beta} e)_a{}^k = \frac{\partial \mathcal{L}'}{\partial(\partial_\mu g^{ij})} (S^{\alpha\beta} g)^{ij}. \tag{40}$$

Here the $S^{\alpha\beta}$ are the antisymmetric infinitesimal generators of Lorentz transformation, acting as

$$\begin{aligned} (S^{\alpha\beta} e)_a{}^k &= \eta^{\alpha k} e_a{}^\beta - \eta^{\beta k} e_a{}^\alpha \\ (S^{\alpha\beta} g)^{ij} &= (\eta^{\alpha i} g^{\beta j} - \eta^{\beta i} g^{\alpha j}) + (\eta^{\alpha j} g^{i\beta} - \eta^{\beta j} g^{i\alpha}). \end{aligned} \tag{41}$$

The BR tensor does not depend on the choice of the configurations variables, since only Noether currents are involved. For Lagrangians containing second derivatives Bak *et al* [8] gave an improved formula.

We now prove a first central result. The BR tensor is related to the gravitational field equations by

$$T_{BR}{}^k{}_l = -e_a{}^k \left(\frac{\partial \mathcal{L}}{\partial e_a{}^l} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i e_a{}^l)} \right) = -2 g^{kj} \left(\frac{\partial \mathcal{L}}{\partial g^{jl}} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i g^{jl})} \right) \tag{42}$$

where the second identity applies only if we do not have spin-$\frac{1}{2}$ matter present and can express the Lagrangian through the metric.

In order to prove (42) we first calculate the spin part $h^{\mu\alpha\beta}$ and then use (35) for the variation. We have

$$\begin{aligned} L^{\mu\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial \Omega^a{}_{bc}} \frac{\partial \Omega^a{}_{bc}}{\partial(\partial_\mu e_d{}^k)} (S^{\alpha\beta} e)_d{}^k \\ &= \frac{\partial \mathcal{L}}{\partial \Omega^a{}_{bc}} e^a{}_k \{\eta^{\beta k}(e_b{}^\mu e_c{}^\alpha - e_c{}^\mu e_b{}^\alpha) - \eta^{\alpha k}(e_b{}^\mu e_c{}^\beta - e_c{}^\mu e_b{}^\beta)\} \end{aligned} \tag{43}$$

and calculate $h^{\mu\alpha\beta}$ according to (39) to

$$h^{\mu\alpha\beta} = -\frac{\partial \mathcal{L}}{\partial \Omega^a{}_{bc}} e^a{}_k \eta^{\alpha k} (e_b{}^\mu e_c{}^\beta - e_c{}^\mu e_b{}^\beta) \tag{44}$$

where we made use of the antisymmetry of Ω in the lower indices. Next we calculate $e_a{}^k(\frac{\partial \mathcal{L}}{\partial e_a{}^l} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i e_a{}^l)})$. Ignoring matter fields for a moment, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial e_a{}^l} &= \Lambda \frac{\partial e}{\partial e_a{}^l} + e \frac{\partial \Lambda}{\partial \Omega^d{}_{bc}} \frac{\partial \Omega^d{}_{bc}}{\partial e_a{}^l} \\ &= -e^a{}_l \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} [e^d{}_i (\delta^a_c \partial_l e_b{}^i - \delta^a_b \partial_l e_c{}^i) - e^a{}_i e^d{}_l (e_c{}^j \partial_j e_b{}^i - e_b{}^j \partial_j e_c{}^i)] \end{aligned} \tag{45}$$

and

$$\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i e_a{}^l)} = \partial_i \left[\frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} e^d{}_l (\delta^a_c e_b{}^i - \delta^a_b e_c{}^i) \right]. \tag{46}$$

Combining (45) and (46) we obtain

$$\begin{aligned}
e_a{}^k \left(\frac{\partial \mathcal{L}}{\partial e_a{}^l} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i e_a{}^l)} \right) &= -\delta^k_l \mathcal{L} + e_a{}^k \frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} [e^d{}_i (\delta^a_c \partial_l e_b{}^i - \delta^a_b \partial_l e_c{}^i) \\
&\quad - e^a{}_i e^d{}_l (e_c{}^j \partial_j e_b{}^i - e_b{}^j \partial_j e_c{}^i)] - e_a{}^k \partial_i \left[\frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} e^d{}_l (\delta^a_c e_b{}^i - \delta^a_b e_c{}^i) \right] \\
&= -\delta^k_l \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} [e^d{}_i (e_c{}^k \partial_l e_b{}^i - e_b{}^k \partial_l e_c{}^i)] - \partial_i \left[\frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} e^d{}_l (e_c{}^k e_b{}^i - e_b{}^k e_c{}^i) \right] \\
&= -\delta^k_l \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Omega^d{}_{bc}} \frac{\partial \Omega^d{}_{bc}}{\partial(\partial_k e_a{}^i)} \partial_l e_a{}^i + \partial_i h^k{}_l{}^i = -T_{BR}{}^k{}_l.
\end{aligned} \tag{47}$$

When we include matter fields we get the additional term

$$e_a{}^k \frac{\partial \mathcal{L}}{\partial(e_a{}^j \partial_j \Phi)} \delta^a_d \partial_l \Phi = \frac{\partial \mathcal{L}}{\partial(\partial_k \Phi)} \partial_l \Phi \tag{48}$$

from the variation with respect to $e_a{}^l$, which is the correct contribution of the matter fields to the BR-tensor since there is no contribution to the spin part from the matter fields due to their behaviour as scalar under coordinate transformations. This completes the proof of (42).

An immediate consequence from (42) is that the characteristic properties of the BR-tensor, its symmetry $T_{BRl}{}^k = T_{BR}{}^k{}_l$ and conservation property $\partial_k T_{BR}{}^k{}_l = 0$, also follow from the gravitational field equations. From (42) and the known result of the variation of the Einstein–Hilbert Lagrangian with respect to g, the BR-tensor for gravitation alone is given by

$$T_{BR}{}^k{}_l = -\frac{c^4}{8\pi\kappa} \sqrt{-g} G^k{}_l. \tag{49}$$

For the case of spin-$\frac{1}{2}$ matter we have reproduced the less evident result that the source for the gravitational field is a symmetry tensor. Here the symmetry is a consequence of the matter field equations [14], and not *a priori* valid as in the case of classical and spin-0 matter.

When we extend our result to the case with the electromagnetic field, we have to be careful, since the action

$$S_{EM} = \frac{1}{4\mu_0 c} \int \mathrm{d}^4x \, \sqrt{-g} g^{kl} g^{k'l'} F_{kl} F_{k'l'} \tag{50}$$

is not of the form of (35). But the common energy-momentum tensor for the electromagnetic field is the Hilbert tensor, and can be obtained by varying the Lagrangian in (50) with respect to $g_{\mu\nu}$ [5]. Only that, in this case, its form is not immediately identical with the BR-tensor for this field, since the electromagnetic field equations are used to derive this result. We will see in the next section that in order to avoid meaningless null results, it is necessary to distinguish between identities which hold for all fields and those which hold only on-shell.

3. Canonical energy-momentum tensor density class

3.1. Formal aspects

We know that we need not fix a Lagrangian uniquely in order to get unique Euler–Lagrange equations. To any Lagrangian $\mathcal{L}_0$ defined on a four-dimensional manifold with a given set of coordinates, we may add a surface term $\mathcal{L}'$,

$$\mathcal{L}_0 \to \mathcal{L}_0 + \mathcal{L}' \tag{51}$$

where $\mathcal{L}'$ has the form

$$\mathcal{L}' = \partial_\mu B^\mu \tag{52}$$

in any of the given set of coordinates; we call this an ordinary divergence. The structure of the surface term has to be preserved under coordinate transformations. We give two simple examples. If we are on $\boldsymbol{R}^4$ and have all Poincaré transformations as coordinate transformations available, the form of a conventional divergence is preserved under these. On the other hand, if we are in general relativity with the whole diffeomorphism group available, any term of the kind

$$\mathcal{L}' = \sqrt{-g}\mathrm{D}_i V^i \tag{53}$$

can be written as a conventional divergence in any coordinates if V is a general covariant vector [1].

The Euler–Lagrange equations obtained from $\mathcal{L}_0$ and $\mathcal{L}_0 + \mathcal{L}'$ are identical. So, in order to derive the field equations, we need to know only the class of Lagrangians to which $\mathcal{L}_0$ belongs, where the class is defined as an equivalence class of functionals on the fields, with equivalence up to the addition of a conventional divergence. It is obvious that we cannot expect that the CEM tensors derived from different elements such as a class of Lagrangians are identical. But it is straightforward to show that any ordinary divergence leads to an identically conserved tensor. By which we mean any functional H^μ_ν on the fields with $\partial_\mu H^\mu_\nu = 0$ for all fields, and not only for those which satisfy the field equations. Tensors of the form $H^\mu_\nu = \partial_\lambda h^{\lambda\mu}_\nu$ with h antisymmetric in λ and μ are the standard examples. It is less obvious that the BR tensor obtained from a surface term does not vanish. This follows from comparison of the result obtained from Bak *et al* [8] with ours. Both their (29) and our (30) Lagrangian differ from the Einstein–Hilbert Lagrangian by a surface term. We obtained the Einstein tensor as BR tensor, whereas they obtained in addition an identically conserved tensor, their (32), in which the Einstein tensor has been eliminated with the help of the field equations [13]. So two aspects become clear. First, the improved BR construction [8] does not lead to a unique energy-momentum tensor for a class of Lagrangians, second we must avoid using the field equations in order to avoid meaningless null results. The first point leads up to define the energy-momentum tensor only up to an identically conserved one, so that the divergence will be the same for the whole class of these tensors, the second point leads us to define the elements of this class as functionals on the whole space of fields, not only on-shell. This means that we exclude the field equations for formal manipulations, with the advantage that we can avoid null results and moreover identify the individual parts of the tensors that belong to particles or the various fields, respectively. Using the field equations would mix these parts.

For a given class of Lagrangians, we finally define the conserved energy momentum as that class of tensor-valued functionals on the whole space of field configurations to which the CEM tensor belongs. Equivalence here is ‘up to the addition of an identically conserved tensor’. In the case of the Einstein–Hilbert action many known examples belong to this class: the Einstein tensor, the BR-tensor, all pseudo EM tensors. We generally assume that any representative of the given class of Lagrangians can be written as a sum of a particle or matter Lagrangian, $\mathcal{L}_M$, an electromagnetic field part, $\mathcal{L}_{EM}$, and a gravitational part $\mathcal{L}_G$. Not all of these parts need to be present, we also consider cases where some fields are external, non-dynamical background fields not subject to variation.

3.2. Classical point particles in external fields

We look at a classical point particle in external fields with action

$$S_{M,cl} = \int \mathrm{d}\tau \left[\frac{m}{2}\dot{x}^k(\tau)\dot{x}^l(\tau)g_{kl}(x(\tau)) + q\dot{x}^k(\tau)A_k(x(\tau))\right] \tag{54}$$

which extends to (33) to include the electromagnetic coupling. The fields g_{kl} and A_k are considered as external, non-dynamical background fields which are not subject to variation. In general we will not have a conserved current, but if we can find a coordinate system in which we have translation invariance with respect to one coordinate, we may apply the Noether theorem to obtain the corresponding conserved current. A common example is energy conservation for time-independent fields: if $\partial_0 g = \partial_0 A = 0$ then $\dot{p}_0 = \mathrm{d}(\delta L/\delta\dot{x}^0)/\mathrm{d}\tau = 0$. The (conserved) energy-momentum tensor for (54) is given by

$$T_{C,M}{}^k{}_l(x) = \int \mathrm{d}\tau\, \frac{\partial\mathcal{L}}{\partial\dot{x}^l}\dot{x}^k\delta^4(x - x(\tau)) \tag{55}$$

since the property $\dot{p}_l = 0$, $p_l = \delta\mathcal{L}/\delta\dot{x}^l$ is equivalent to $\partial_k T_{C,M}{}^k{}_l(x) = 0$. With standard rules for delta-functions we carry out the τ-integration in (55) and obtain

$$T_{C,M}{}^k{}_l(x) = p_l(t)v^k(t)\delta^3(x^i - x^i(t)) \tag{56}$$

where $v^k(t) = \mathrm{d}x^k(t)/\mathrm{d}t$, $v^0 = c$. Integration over spacelike volume V which contains the particle yields the energy

$$E = \int_V \mathrm{d}^3x\, T_{C,M}{}^0{}_0(x) = p_0 c. \tag{57}$$

This energy is not determined uniquely when we consider the class of Lagrangians to which (54) belongs. Surface terms may lead to the addition of a constant; any constant tensor is identically conserved. For different gauges of the electromagnetic field we obtain different values of the energy. Since the fields are external we do not have any compensating part from the electromagnetic field energy. Choosing a time-dependent gauge function could even mean that we lose energy conservation.

A further point to stress is the structure of the tensor (55). We cannot raise the covariant momentum index with the metric without losing the correct expression for the energy. If we look at the value of p_0 in a weak-field approximation with $g_{00} = 1 + 2\Phi/c^2$ we have [15]

$$p_0 c \simeq mc^2 + m\Phi + qcA_0 \tag{58}$$

which depends in the expected way on the potentials Φ and A_0. If we multiplied the energy density by g^{00} in order to obtain two contravariant indices on the density we would get

$$E' = \int_V \mathrm{d}^3x T_{C,M}{}^{00}(x) = g^{00}p_0 c \simeq mc^2 - m\Phi + qcA_0 \tag{59}$$

with unphysical dependence on the gravitational potential Φ. For the same reason we may not lower the contravariant current index of $\dot{x}^{\prime}$ or divide by $\sqrt{-g} \neq 1$. Nor may we replace

$p_0 c$ by $mc\dot{x}_0$, whence we would lose the dependence of the energy on the electromagnetic potential A_0. This shows that the conserved energy-momentum tensor in given by (56), up to identically conserved tensors. The types of the indices are uniquely determined, we have a covariant momentum index, and a contravariant current index. The factor $\sqrt{-g}$ must be included, so that in general relativity energy-momentum transforms as a tensor density. So we have shown the weak-field limit fixes the conserved energy-momentum tensor for a classical point particle to the CEM tensor density class.

3.3. Quantum matter in external fields

The considerations for classical particles carry over to spin-0 and spin-$\frac{1}{2}$ quantum matter straightforwardly. We consider the actions (33) for particles in external fields, with the electromagnetic field minimally coupled in addition. The corresponding CEM tensors are

$$T_{C,M}{}^{k}{}_{l}(x) = \delta^{k}{}_{l}\mathcal{L}_M + \begin{cases} \dfrac{1}{m} e\Re[\eta^{ab}(e_a{}^{k}(\partial_k + \mathrm{i}qA_k)\phi^*)(e_b{}^{k}\partial_l\phi)] & \text{(spin 0)} \\ e\Re[\bar{\psi}\gamma^a \mathrm{i}\hbar e_a{}^{k}\partial_l\psi] & \text{(spin } \tfrac{1}{2}\text{)}. \end{cases} \tag{60}$$

The Lagrangian $\mathcal{L}_M$ vanishes on-shell, but not as a functional of arbitrary wavefunctions. We observe that these tensors are not identical with the corresponding Hilbert tensors, which are obtained by variation with respect to the tetrad. This holds only in the absence of an electromagnetic field. The CEM tensors (60) in general are not symmetric. We need compensating spin parts from the field Lagrangians $\mathcal{L}_G$ and $\mathcal{L}_{EM}$ as well as the field equations if we want to achieve symmetry. The BR tensor together with these parts will form the Hilbert tensor. The BR tensor obtained from the matter Lagrangian in external fields need not be symmetric, since the proof of symmetry needs the full Poincaré invariance [3], which need not be present there. So the CEM tensor is the only systematic construction to obtain conserved energy or momentum currents for matter in external fields. It is the only construction which can be applied in the presence of only partial symmetries, with the conserved currents being those obtained from the Noether theorem.

3.4. Matter and electromagnetic fields

We add the action (50) of the electromagnetic field to the matter actions and keep the metric fixed as a background field. The CEM tensor for the electromagnetic field is given by

$$T_{C,EM}{}^{k}{}_{l} = \frac{\sqrt{-g}}{\mu_0}\left(\tfrac{1}{4}\delta^{k}{}_{l}F^{ij}F_{ij} - F^{kj}\partial_l A_j\right). \tag{61}$$

If the metric is invariant under translations in the lth coordinate, then the lth component of the total energy momentum is conserved,

$$\partial_k(T_{C,EM}{}^{k}{}_{l} + T_{C,M}{}^{k}{}_{l}) = 0. \tag{62}$$

The CEM tensor is not gauge-invariant, but the on-shell BR tensor is. For the action (50) the spin contribution to the BR tensor is given by [6].

$$T_{S,EM}{}^{k}{}_{l} = \partial_i(\sqrt{-g}F^{ki}A_l). \tag{63}$$

On-shell this can be written as

$$\begin{aligned} T^{os}_{S,EM}{}^{k}{}_{l} &= A_l\partial_i(\sqrt{-g}F^{ki}) + \sqrt{-g}F^{ki}\partial_i A_l \\ &= A_l j^k_M + \sqrt{-g}F^{ki}\partial_i A_l \end{aligned} \tag{64}$$

with the j_M^k the electrical current of the matter part, $j_M^k = \partial\mathcal{L}_M/\partial A_k$. If we add the first summand on the r.h.s. of (64) to the matter CEM tensor, the second to the electromagnetic CEM tensor, we obtain the usual combined, gauge invariant, energy-momentum tensor, which is also the Hilbert tensor.

3.5. Gravity included

3.5.1. Field equations. As we have seen in section 2, the BR-tensor derived from the gravitational part of the Lagrangian can be split into two parts, the CEM tensor ${T_C}^k{}_l$ and the spin part $\partial_i {h^k}_l{}^i$. Since ${h^k}_l{}^i$ is antisymmetric in the first and third indices, it is identically conserved, which means that

$$\partial_k \partial_i {h^k}_l{}^i = 0 \qquad (65)$$

holds irrespective of whether the fields are solutions of the equations of motion or not. The free gravitational field equations are brought into the form

$$\partial_i {h^k}_l{}^i = {T_{C,G}}^k{}_l. \qquad (66)$$

Here we should make a remark on the sign conventions. We chose the Landau and Liftschitz [5] time-like convention $+---$ for g. All actions are chosen such that $\delta S/\delta g^{kl} \sim -\frac{1}{2c} T^{kl}$ with positive energy density $T^0{}_0 > 0$ and $T_{00} > 0$. Therefore, the Einstein field equations coupled to matter and electromagnetic field have the form

$$\frac{c^4}{8\pi\kappa}\sqrt{-g}\,G^k{}_l = ({T_M}^k{}_l + {T_{EM}}^k{}_l) \qquad (67)$$

and we have a positive trace $T = -R$. In the space-like sign convention $-+++$ for g as used by Misner *et al* [1] $T^0{}_0$ is negative, so their $\sqrt{-g}G^k{}_l = 8\pi\kappa c^{-4} T^k{}_l$ leads to a positive trace $R = -T$. Since R depends on the sign convention, changing its sign with that of g, we see that conventions used by Landau and Lifschitz [5] and Misner *et al* [1] are equivalent. This 'Einstein sign' [1] is not a question of convention, since taking a negative sign on the r.h.s. of (67) leads to a change of sign of the source mass in the generated field, e.g. the external Schwarzschild solution leading to repulsive gravitational forces.

The considerations of section 3.1 have shown that we may use the electromagnetic field equations to bring (67) into the form

$$\partial_i {h^k}_l{}^i - {T_{S,EM}}^k{}_l = ({T_{C,G}}^k{}_l + {T_{C,M}}^k{}_l + {T_{C,EM}}^k{}_l) \qquad (68)$$

with an identically conserved l.h.s. and the overall CEM tensor as source, its conservation following from the field equation. Since we do not obtain a spin part from the matter Lagrangians, (68) can equivalently be written as

$$-T_S = T_C \qquad \text{or} \qquad T_{BR} = T_C + T_S = 0 \qquad (69)$$

with overall spin and canonical parts T_S and T_C, respectively.

The form (68) allows us to convert any three-dimensional volume integrals of the energy density over a volume V into a Gaussian flux integral over the surrounding two-surface S, a central idea behind the construction of pseudo energy-momentum tensors [1, 5]. Using (63) we have

$$\int_V \mathrm{d}^3 r\, {T_C}^0{}_0 = \int_V \mathrm{d}^3 r\, \partial_I ({h^0}_0{}^I - \sqrt{-g}F^{0I}A_0) = \int_S \mathrm{d}^2 S_I\, ({h^0}_0{}^I - \sqrt{-g}F^{0I}A_0) \qquad (70)$$

since ${h^0}_0{}^0 = F^{00} = 0$. Here, and further below, the capital Latin indices $I, J, K = 1, 2, 3$ denote the space-like part only, with $+++$ summation convention. The difference to the usual construction is that both sides in (68) are not necessarily symmetric. Usually symmetry is required in view of angular momentum conservation. The canonical Noether current obtained from Lorentz invariance is

$$j_C{}^{ikl} = x^i T_C{}^{lk} - x^k T_C{}^{li} + L^{lik} \tag{71}$$

with canonical spin part

$$L^{lik} = \frac{\partial \mathcal{L}}{\partial(\partial_l \psi)}(S^{ik}\psi) \tag{72}$$

as in (40), but ψ now standing for all fields; S^{ik} are the generators of Lorentz transformations as in section 2.2. The constructions by Belinfante [3] and Rosenfeld [4] showed that one can add an identically conserved current to (71) in order to arrive at the conserved angular momentum density

$$j^{ikl} = x^i T_{BR}{}^{lk} - x^k T_{BR}{}^{li}. \tag{73}$$

This equation is used to prove the symmetry of the BR tensor from angular momentum conservation [3, 4, 6], we have

$$0 = \partial_l j^{ikl} = T_C^{ik} - T_C{}^{ki} + \partial_l L^{lik} = T_{BR}{}^{ik} - T_{BR}{}^{ki}. \tag{74}$$

So the logic is that angular momentum conservation always follows from Lorentz invariance, if we do not have full Lorentz invariance, the BR tensor will not be symmetric. The conservation of energy-momentum follows from translational invariance, independent of Lorentz invariance.

3.5.2. Canonical energy-momentum tensor for gravity Einstein himself used the Noether concept of conservation laws to show that his theory of gravitation conserved overall momentum and energy. The conserved CEM tensor obtained is not symmetric, which is commonly considered to be unsatisfactory. In our concept the symmetry is not a required property for a single tensor. In the class of tensors generated by the CEM tensor we have a conserved symmetric energy-momentum tensor, since the BR tensor is a multiple of the Einstein tensor density, cf relation (49). This tensor is also covariant, but since it vanishes on-shell, the related energy density has no positivity property, which would be sufficient to prove the stability of the theory. So we analyse the CEM tensor obtained from the tetrad Lagrangian $\mathcal{L}_G = (c^3/16\pi\kappa)\tilde{R}$ (30) with respect to positivity. Local positivity of the energy density is not a necessary condition for the stability of the theory, the positivity of the total energy suffices under proper conditions [17].

For the tetrad Lagrangian $\mathcal{L}_G$ the CEM tensor for gravity is given by

$$T_{C,G}{}^k{}_l = c\left(\delta^k{}_l \mathcal{L}_G - \frac{\partial \mathcal{L}_G}{\partial(\partial_k e_a{}^j)}\partial_l e_a{}^j\right). \tag{75}$$

This tensor is purely quadratic in the first derivatives of the tetrad field, containing no second derivatives. We look at this tensor for weak gravitational fields where we set

$$e_a{}^j = \delta_a{}^j - \tfrac{1}{2}\epsilon\delta_a{}^k \eta^{jl} h_{kl} \tag{76}$$

with a symmetric perturbation h_{kl} of the background, so that the metric is

$$g_{kl} = \eta_{kl} + \epsilon h_{kl} = \mathrm{O}(\epsilon^2). \tag{77}$$

We use ϵ as a smallness parameter. When we scale $\kappa \to \kappa\epsilon$ and insert this together with (76) into the field equation (68) we obtain the usual linearized field equation in order $\mathrm{O}(\epsilon^0)$. This equation implies that the dominant part of the positive particle and electromagnetic field energy is compensated by the gravitational spin part when the sum of spin part and canonical part vanishes. Since the spin part is separately conserved, no exchange of energy can occur to or from particles or electromagnetic fields at this order. This already means that particles cannot be created to annihilated at the cost of gravitational energy, the theory is stable in the lowest order.

The lowest order ϵ contribution from the canonical energy-momentum tensor is of order $O(\epsilon)$, given by

$$T^{(1)}_{C,G}{}^k{}_l = \frac{c^4}{64\pi\kappa}(2\delta^c_a\omega^{db}{}_d - \tfrac{1}{2}\omega_a{}^{bc} - \omega^b{}_a{}^c)(\delta^k_c\partial_l h_b{}^a - \delta^k_b\partial_l h_c{}^a - \tfrac{1}{2}\delta^k_l\omega^a{}_{bc}) \tag{78}$$

where

$$\omega^a{}_{bc} = \partial_c h_b{}^a - \partial_b h_c{}^a \tag{79}$$

is the linearization of $2\Omega^a{}_{bc}$ with respect to ϵ. All indices are raised or lowered with the help of η. Since the energy density $T^{(1)}_{C,G}{}^0{}_0$ is not manifestly positive, we look at special cases.

We first consider gravitational waves. In the transverse-traceless gauge [1, section 35.4], characterized by $h_{i0} = 0$, $h_{II} = 0$, $\partial_J h_{IJ} = 0$, (78) reduces to

$$T^{(1)TT}_{C,G}{}^0{}_0 = \frac{c^4}{64\pi\kappa}[3(\partial_0 h_{IJ})^2 - \tfrac{1}{2}(\partial_I h_{JK} - \partial_J h_{IK})^2]. \tag{80}$$

When we insert plane waves, we see that the contributions from the terms $(\partial_0 h_{IJ})^2$ and $-\frac{1}{2}(\partial_I h_{JK} - \partial_J h_{IK})^2$ in (80) are the same, so that this energy density becomes, when averaged over a couple of wavelengths, identical with the positive one given by Misner *et al* [1, (35.23)].

But the problem arises that this energy of periodic plane waves cannot be considered an approximation to the energy of exact solutions, since any exactly periodic solution of the vacuum field equation (66) has zero mean energy, for the following reason, If the tensor $h^k{}_l{}^i$ is a periodic function of $\omega t - \boldsymbol{k}\cdot\boldsymbol{x}$ as a single variable, then it allows for a Fourier decomposition

$$h^k{}_l{}^i(ct - \boldsymbol{k}\cdot\boldsymbol{x}) = h^k_{0l}{}^i + \sum_{n=1}^{\infty}(a^k_{nl}{}^i \sin n(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}) + b^k_{nl}{}^i \cos n(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})). \tag{81}$$

Taking derivatives eliminates the contribution from the constant coefficients $h^k_{0l}{}^i$, so that the average of $\partial_i h^k{}_l{}^i$ over any domain of periodicity vanishes. Hence it follows from (66) that the averaged canonical energy density of periodic solutions of the Einstein field equations vanishes. This remains true if any identically conserved tensor of the form $\partial_i H^k{}_l{}^i$ is added to the CEM tensor. This implies that gravitational waves can carry energy only to the degree that they deviate from periodicity, but they do not necessarily do so. For example, exact plane-waves pulses can be constructed [16] where the space-time manifold is flat outside the wave pulse, which means that there is no total energy producing space–time curvature outside the pulse [1, section 35.9]; this can be immediately shown using the surface integral (70). This result implies that a system of matter cannot lose energy by emission of periodic waves or zero-energy wave pulses, it cannot gain energy when interacting with such waves.

For static fields we obtain an energy density from the CEM tensor that is negative if $h_{I0} = 0$ and $\partial_0 h_{kl} = 0$. The first-order contribution can be written as

$$T^{(1)static}_{C,G}{}^0{}_0 = \frac{c^4}{64\pi\kappa}\Bigg[-\tfrac{1}{2}(\partial_\alpha h_{\beta\beta})^2 - \tfrac{7}{4}(\partial_\beta h_{\alpha\beta} - \partial_\alpha h_{\beta\beta})^2 - \tfrac{5}{2}(\partial_\alpha h_{00} - \tfrac{2}{5}\partial_\alpha h_{\beta\beta})^2 - \tfrac{3}{4}\sum_{\alpha\neq\beta\neq\gamma}(\partial_\alpha h_{\beta\gamma} - \partial_\beta h_{\alpha\gamma})^2\Bigg] \tag{82}$$

which is manifestly negative. This means that the exact canonical energy density will be negative as long as (82) can be considered a good approximation, as for weak, slowly varying fields.

As an example we calculate the energy density of the exterior Schwarzschild solution in isotropic coordinates from the full tensor (75) to

$$T^{SS}_{C,G}{}^{0}{}_{0} = -\frac{c^4}{64\pi\kappa}\frac{M^2}{2r^4}\left(1+\frac{M}{2r}\right)^8 \qquad M = \frac{m\kappa}{c^2},\ r > \frac{M}{2} \tag{83}$$

which is valid outside any static, spherically symmetric matter distribution. The density (83) is negative everywhere outside the matter, but the total energy $\int_{\boldsymbol{R}^3} \mathrm{d}^3 r\, T_C{}^0{}_0$, which includes the contribution from the matter, is not, since from (70) we have

$$\int_{\boldsymbol{R}^3} \mathrm{d}^3 r\, T_C{}^0{}_0 = \lim_{R\to\infty}\int_{S_R} \mathrm{d}^2 S_I\, h^0{}_0{}^I = \lim_{R\to\infty}\int_{S_R} \mathrm{d}^2 S_I\, \frac{c^4}{16\pi\kappa}(\partial_J h^N_{IJ} - \partial_I h^N_{JJ}) = mc^2 \tag{84}$$

where S_R is the sphere of radius R, and we could make use of the fact that in the limit $R \to \infty$ we need only the leading Newtonian contribution to $h^0{}_0{}^I$ (see (44)), with $h^N_{IJ} = \delta_{IJ} 2M/r$. The Newtonian approximation in (84) agrees with the standard Arnowitt–Deser–Misner (ADM) expression at infinity [1, section 20.1, 18], hence our theory leads to the same results regarding positivity of the total energy for solutions of the Einstein field equations in asymptotically flat pseudo-Euclidean space-times without singularities [17]. The result (84) does not hold for black holes, where we have a total energy $\int_{r>M/2} \mathrm{d}^3 r\, T_{C,G}{}^0{}_0 = -511mc^2/144$ outside the horizon, but in the interior we do not have translational invariance with respect to a time-like coordinate that would allow us to define a Noether energy current. We finally note that the result (84) pertains to the case of exterior Kerr–Newman geometries [1, section 33.2], so that the total energy of any symmetric massive, charged, and spinning system of matter and fields without singularities is identical with the energy of a corresponding non-interacting, fieldless system. The contributions from the gravitational and electromagnetic fields are effectively zero.

4. Summary

Since most of the many energy-momentum tensors for gravity suggested so far cannot be rejected on an experimental basis, we collected them in a class of tensors, where members differ only by identically conserved tensors. This was motivated by the observation that the physical evolution equations of any system are uniquely defined through a class of Lagrangians where different members differ by surface terms, but lead to different conserved Noether currents, since a surface term may give rise to an identically conserved Noether current. In each case the class of energy-momentum tensors, in the case of general relativity we have tensor densities, is generated by the canonical energy-momentum tensor corresponding to the Lagrangian under consideration. This concept was shown to be reasonable also in those cases where we do not have full Poincaré symmetry, as for particles in external fields. In general, each one-parameter symmetry group leads to a conserved Noether current, in case of a class of Lagrangians to a class of these, irrespective of whether other symmetries are realized. Only when we have full Poincaré symmetry of the Lagrangian, or class of Lagrangians, can we employ the BR construction to find a symmetric energy-momentum tensor. We showed that this construction also depends on chosen surface terms. With the concept of a class of energy-momentum tensors we do not have an absolute energy, each member is equally suited to calculate the predictions for the outcome of some experiment on the basis of given initial kinematic data. On the other hand, this means that we are always free to choose some specific Lagrangian for special purposes, such as when we need a Lagrangian without second derivatives as a starting point for the construction of a Hamiltonian formulation. When more than one such Lagrangian exists, we have a choice among equivalent Hamiltonians. The tetrad Lagrangian of gravity (30), on which we based our analysis, is of this type, and we found that the corresponding BR tensor is precisely a multiple of the Einstein-tensor density.

Usually positivity of the energy density is considered in a desirable property, since it entails the stability of the system. With respect to gravity, no such tensor has been found so far, but here positivity of the total energy suffices for stability. In the case of the tetrad Lagrangian, the energy density of the BR tensor is null in vacuum, and even negative inside matter or electromagnetic fields, since the Einstein field equations are equivalent to the vanishing of the combined BR tensor for gravity, matter and electromagnetic field. So we looked at the canonical energy density, and showed that in general it is not positive. For periodic solutions of the Einstein field equations it has mean zero, which implies that the emission and absorption of gravitational waves must be treated as a dynamical problem of the full nonlinear theory. For static solutions in vacuum the energy density is locally negative under quite general assumptions, but for the total energy we obtained the standard ADM result.

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